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Pair creation in Feshbach–Villars formalism with two components

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Abstract. We have studied the behavior of the Feshbach–Villars equation (FV_0) in comparison with the Klein–Gordon one (KG) in the problem of particle pair creation from the vacuum in an external electromagnetic field, considering two approaches: the Schwinger effective action method and the Bogoliubov transformation technique. In the first approach the vacuum to vacuum transition amplitude is calculated treating the FV_0 field like a bosonic field. For the second approach, that uses canonical quantization, it is shown that the relative fields and their conjugate moments obey a commutation relation and not anti-commutation one. The pair creation probability and the probability that the vacuum remains a vacuum calculated from the FV_0 equation are, consequently, the same as those obtained from the KG one.

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1 Introduction

Generally, there exist in relativistic quantum mechanics two well-known wave equations: the first order Dirac equation which describes the spin $\frac{1}{2}$ particle and the second order Klein–Gordon (KG) equation that used to describe the spin 0 particle. However, the latter equation, due to the second order derivative, involves a main difficulty, namely that the probability density

$$\rho_{\mathrm{KG}} \propto \varphi^* \frac{\partial}{\partial t}\varphi - \varphi \frac{\partial}{\partial t}\varphi^* \tag{1}$$

is not positive definite [1]. In order to restore this problem and to get a positive density of probability Feshbach and Villars [2] have formulated, for a spinless particle, another first order wave equation, called FV_0, that has a certain resemblance to Dirac's one and to the non-relativistic Schrödinger equation where the KG wave function φ and its time-derivative are replaced by the two-component FV_0 vector [2,3]

$$\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi + \frac{\mathrm{i}}{m}\left(\frac{\partial}{\partial t} + \mathrm{i}eV(x)\right)\varphi \\ \varphi - \frac{\mathrm{i}}{m}\left(\frac{\partial}{\partial t} + \mathrm{i}eV(x)\right)\varphi \end{bmatrix}. \tag{2}$$

In the FV_0 two-component formulation, the equation of motion has the following Hamiltonian form:

$$\mathrm{i}\frac{\partial}{\partial t}\psi_{\mathrm{FV}} = H_{\mathrm{FV}}\psi_{\mathrm{FV}}, \tag{3}$$

where the Hamiltonian

$$H_{\mathrm{FV}} = -\frac{1}{2m}\left(\boldsymbol{\nabla} - \mathrm{i}e\boldsymbol{A}\right)^2(\tau_3 + \mathrm{i}\tau_2) + m\tau_3 + eV(x), \tag{4}$$

which carries the isospin Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

is pseudo-Hermitian, $H^+_{\mathrm{FV}} = \tau_3 H_{\mathrm{FV}} \tau_3$, and the continuity equation can be written in the form

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\boldsymbol{J} = 0, \tag{6}$$

where the density ρ and the current $\boldsymbol{J}$ are given by

$$\rho = e\overline{\psi}\psi \tag{7}$$

and

$$\boldsymbol{J} = \frac{e}{2\mathrm{i}m}\Big[\overline{\psi}(\tau_3 + \mathrm{i}\tau_2)\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\overline{\psi}(\tau_3 + \mathrm{i}\tau_2)\psi - \frac{e}{m}\boldsymbol{A}\,\overline{\psi}(\tau_3 + \mathrm{i}\tau_2)\psi\Big], \tag{8}$$

with

$$\overline{\psi} = \psi^+\tau_3. \tag{9}$$

It is easy to show that ρ is positive for positive energy and negative for negative energy, so that it is interpreted as the charge density, and positive and negative solutions are interpreted respectively as particle states and antiparticle ones. Note that those states are connected to one another by a charge conjugation transformation defined by

$$\psi \to \psi_c = \tau_1\psi^*. \tag{10}$$

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If ψ is a solution of the equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi = H_{\mathrm{FV}}\ (e)\,\psi, \tag{11}$$

then ψ_c is a solution of the equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi_c = H_{\mathrm{FV}}\ (-e)\,\psi_c.$$

Let us mention, in the end of this reminder, that the expectation value of an operator $\mathcal{O}$ is defined as

$$\langle\mathcal{O}\rangle = \int \overline{\psi}\mathcal{O}\psi\mathrm{d}^3x, \tag{12}$$

and the classical equation of motion reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathcal{O}\rangle = \mathrm{i}\,\langle[H_{\mathrm{FV}},\mathcal{O}]\rangle + \left\langle\frac{\partial}{\partial t}\mathcal{O}\right\rangle. \tag{13}$$

During the last decades, many authors have been interested in the Hamiltonian form of the relativistic wave equation constructed from the two-component wave function (FV_0) for a spinless particle and from the eight-component wave function ($\mathrm{FV}_{\frac{1}{2}}$) for spin 1/2 particles. A great effort has been made to compare the behavior of the FV_0 and $\mathrm{FV}_{\frac{1}{2}}$ formulations to the KG and Dirac ones in solving physical problems either by using a path-integral approach [4] or by solving the wave equations directly [5, 6]. Recently, the authors of [7] have studied the Compton scattering problem in the $\mathrm{FV}_{\frac{1}{2}}$ formalism and they found that the cross section is given, like in the Dirac equation, by the Klein–Nishina formula.

In this paper, we shall study, by using the FV_0 two-component theory, the problem of pair creation from the vacuum in an external electromagnetic field which remains, actually, a very important topic. The subject of spontaneous particle creation has more significant implications in quantum field theory (QFT). Namely, the possibility of particle creation, in the presence of an external field, makes the question of single particle quantization more complicated. However, if we take into account that the vacuum state at time $t_i = -\infty$ differs from the vacuum state at time $t_f = +\infty$ we can, in a natural way, handle this difficulty by introducing the "in" and "out" vacuum and particle states. In this context, our work takes on two approaches: the first one is based on the vacuum to vacuum transition amplitude defined from the Schwinger effective action and the second one makes use of a Bogoliubov transformation connecting the "in" and "out" particle states.

In the first stage we shall demonstrate that in the FV_0 formulation the vacuum to vacuum transition amplitude is given, like in the KG equation, by a simple determinant of the operator associated with the wave equation. In the second way we shall show, after quantization of the FV_0 field, how to find "in" and "out" states like in the KG equation and we shall illustrate the choice of those states for a smooth potential. Then we shall derive the well-known pair creation probability like in the KG formulation.

2 Vacuum to vacuum transition amplitude

Particle–antiparticle pair creation was first studied by Schwinger many years ago [8]. In effect, he has defined the vacuum to vacuum transition amplitude $\mathcal{A}(\text{vac–vac})$ by an intermediate effective action S_{eff} calculated either by using the proper time method [8] or by using perturbation theory methods [9]. We have

$$\mathcal{A}(\text{vac–vac}) = \exp(\mathrm{i}S_{\text{eff}}); \tag{14}$$

he showed that the pair creation probability can be extracted from the imaginary part of the effective action:

$$\mathcal{P}_{\text{Creat.}} = 1 - |\mathcal{A}(\text{vac–vac})|^2 \simeq 2\mathrm{Im}S_{\text{eff}}. \tag{15}$$

Having shown how to find the vacuum to vacuum transition amplitude $\mathcal{A}(\text{vac–vac})$ and the probability of pair creation, the author of [10] studied this subject in the context of strong field pair production.

To find the amplitude $\mathcal{A}(\text{vac–vac})$ in the FV_0 formulation let us, to begin, recall, briefly, how to define $\mathcal{A}(\text{vac–vac})$ in non-relativistic quantum mechanics as well as in quantum field theory.

In non-relativistic quantum mechanics, the propagator or the transition amplitude from the initial state x_i to the final state x_f for a physical system governed by the Hamiltonian H and the Lagrangian $\mathcal{L}$ is defined, in configuration space, by a matrix element of the evolution operator

$$\mathcal{K}\left(x_f, t_f; x_i, t_i\right) = \left\langle x_f\left|\exp\left(-\mathrm{i}H\left(t_f - t_i\right)\right)\right| x_i\right\rangle, \tag{16}$$

which admits the following functional integral representation:

$$\mathcal{K}\left(x_f, t_f; x_i, t_i\right) = \int \mathcal{D}x(t)\exp\mathrm{i}S\left[x(t)\right], \tag{17}$$

where the action $S\left[x(t)\right]$ is a functional of the continuous trajectory $x(t)$ connecting space-time point (x_i, t_i) with (x_f, t_f);

$$S\left[x\right] = \int_{t_i}^{t_f}\mathrm{d}t\mathcal{L}(x,\dot{x}). \tag{18}$$

Starting from the spectral decomposition of the transition amplitude

$$\mathcal{K}\left(x_f, t_f; x_i, t_i\right) = \sum_n \phi_n(x_f)\phi_n^*\left(x_i\right)\mathrm{e}^{-\mathrm{i}E_n(t_f - t_i)}, \tag{19}$$

it is well known that we can find the wave functions $\phi_n(x)$ as well as the corresponding energies E_n relative to the Hamiltonian H. However, in order to find the vacuum to vacuum transition amplitude $\mathcal{A}(\text{vac–vac})$ we take only the contribution of the ground-state $\phi_0(x)$ corresponding to the lowest energy E_0,

$$\phi_0(x_f)\phi_0^*\left(x_i\right)\mathrm{e}^{-\mathrm{i}E_0(t_f - t_i)}. \tag{20}$$

After having effected a simple Wick rotation from real time to the imaginary one, $T \to -\mathrm{i}T$, and taking the limit $T = t_f - t_i \to \infty$, one can see that the ground-state contribution dominates in the transition amplitude expression (19). Then, in quantum mechanics, the amplitude $\mathcal{A}$(vac–vac) is given by

$$\mathcal{A}(\text{vac–vac}) \overset{T\to\infty}{\approx} \mathcal{N} \int \mathcal{D}x(t) \exp \mathrm{i}S\left[x(t)\right]. \tag{21}$$

Having shown how to extract the vacuum to vacuum amplitude from the Feynman propagator in non-relativistic quantum mechanics, let us derive it in the framework of QFT based on the Lagrangian density

$$\mathcal{L} = \varphi^* \hat{O}_{\mathrm{KG}} \varphi, \tag{22}$$

where the complex field φ describes a relativistic spinless particle of mass m and charge e moving in an external field A_μ and the operator $\hat{O}_{\mathrm{KG}}$ is associated with the KG equation

$$\hat{O}_{\mathrm{KG}} = -\left(\mathrm{i}\partial_\mu - eA_\mu\right)^2 + m^2. \tag{23}$$

We first write the amplitude $\mathcal{A}$(vac–vac) as a functional integral where the summation over all possible trajectories $x(t)$ will be replaced by a sum over all field configurations $\varphi(x)$ and $\varphi^*(x)$; note that $\varphi(x)$ and $\varphi^*(x)$ are considered independents,

$$\begin{aligned}&\mathcal{A}(\text{vac–vac}) \\ &= \int \mathcal{D}\varphi \mathcal{D}\varphi^* \exp\left\{\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}(\varphi, \varphi^*, \partial_\mu\varphi, \partial_\mu\varphi^*)\right\}.\end{aligned} \tag{24}$$

In our second way, in order to calculate this amplitude, we consider the eigenfunctions $\chi_n(x)$ relative to the operator $\hat{O}_{\mathrm{KG}}$,

$$\hat{O}_{\mathrm{KG}} \chi_n(x) = \lambda_n \chi_n(x), \tag{25}$$

forming an orthogonal set

$$\int \mathrm{d}^4x \chi_n(x) \chi_m^*(x) = \delta_{nm}, \tag{26}$$

and we expand

$$\varphi(x) = \sum_n a_n \chi_n(x). \tag{27}$$

One can then find

$$\begin{aligned}&\mathcal{A}(\text{vac–vac}) \\ &= \mathcal{N} \int \prod_i \mathrm{d}a_i \prod_j \mathrm{d}a_j^* \\ &\qquad \times \exp\left\{\mathrm{i}\sum_{n,m} a_n a_m^* \lambda_n \int \mathrm{d}^4x \chi_n(x) \chi_m^*(x)\right\} \\ &= \mathcal{N} \int \mathrm{d}a_i \prod_j \mathrm{d}a_j^* \exp\left\{\mathrm{i}\sum_n |a_n|^2 \lambda_n\right\} \\ &= \mathcal{N}' \prod_n \frac{1}{\lambda_n} = \frac{\mathcal{N}'}{\det \hat{O}_{\mathrm{KG}}},\end{aligned} \tag{28}$$

where $\mathcal{N}$, $\mathcal{N}'$ are two normalization constants that can be fixed by comparing with the same problem without interaction. So the calculation of the amplitude $\mathcal{A}$(vac–vac) reduces, simply, to the calculation of a determinant.

Thus, we have pointed out how to determine the amplitude $\mathcal{A}$(vac–vac) both in non-relativistic quantum mechanics and in quantum field theory (in the KG equation). Let us now calculate it in the FV_0 formulation. Starting from the Lagrangian density for the FV_0 field, we have

$$\begin{aligned}\mathcal{L}_{\mathrm{FV}} &= \mathrm{i}\bar{\psi}\dot{\psi} - eV(x)\bar{\psi}\psi - m\ \bar{\psi}\tau_3\psi \\ &\quad + \frac{1}{2m}\left(\boldsymbol{\nabla} - \mathrm{i}e\boldsymbol{A}\right)\bar{\psi}\left(\tau_3 + \mathrm{i}\tau_2\right)\left(\boldsymbol{\nabla} - \mathrm{i}e\boldsymbol{A}\right)\psi,\end{aligned} \tag{29}$$

which has the standard form

$$\mathcal{L}_{\mathrm{FV}} = \bar{\psi} \hat{O}_{\mathrm{FV}} \psi, \tag{30}$$

where the operator $\hat{O}_{\mathrm{FV}}$ is associated with the FV_0 equation

$$\hat{O}_{\mathrm{FV}} = \mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\left(\boldsymbol{\nabla} - \mathrm{i}e\boldsymbol{A}\right)^2 \eta - eV(x) - m\ \tau_3 \tag{31}$$

and $\eta = (\tau_3 + \mathrm{i}\tau_2)$, we define the vacuum to vacuum transition amplitude $\mathcal{A}$(vac–vac) like in KG theory as a functional integral

$$\mathcal{A}(\text{vac–vac}) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\mathrm{i}\int \mathrm{d}^4x \mathcal{L}_{\mathrm{FV}}\right], \tag{32}$$

and we must treat the fields ψ and $\bar{\psi}$ like ordinary scalar variables (bosonic variables) and not like Grassmannian variables in order to get a positive probability. Then we obtain

$$\mathcal{A}(\text{vac–vac}) = 1/\det \hat{O}_{\mathrm{FV}}. \tag{33}$$

Note that it is not easy to calculate $\det \hat{O}_{\mathrm{FV}}$ directly, so that we introduce a charge conjugation. The FV equation being invariant under this operation, the amplitude $\mathcal{A}$(vac–vac) becomes

$$\mathcal{A}(\text{vac–vac}) = 1/\det \hat{O}^c_{\mathrm{FV}}, \tag{34}$$

where $\hat{O}^c_{\mathrm{FV}}$ is the charge conjugate operator of $\hat{O}_{\mathrm{FV}}$:

$$\hat{O}^c_{\mathrm{FV}} = -\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\left(\boldsymbol{\nabla} - \mathrm{i}e\boldsymbol{A}\right)^2 \eta + eV(x)\bar{\psi}\psi - m\ \tau_3. \tag{35}$$

One can then write

$$\begin{aligned}\mathcal{A}(\text{vac–vac}) &= \frac{1}{\sqrt{\det \hat{O}_{\mathrm{FV}} \hat{O}^c_{\mathrm{FV}}}} \\ &= \exp\left[-\frac{1}{2}\mathrm{tr}\ln\left(\hat{O}_{\mathrm{FV}}\hat{O}^c_{\mathrm{FV}}\right)\right].\end{aligned} \tag{36}$$

Taking into account that $\eta\tau_3 + \tau_3\eta = 2$ and $\eta^2 = 0$, the amplitude $\mathcal{A}$(vac–vac) in the FV_0 equation becomes

$$\mathcal{A}(\text{vac–vac}) = \exp\left\{-\frac{1}{2}\mathrm{tr}\ln\left[\hat{I}\hat{O}_{\mathrm{KG}} + \eta\Lambda\left(x, \frac{\partial}{\partial x}\right)\right]\right\}, \tag{37}$$

where $\hat{I}$ is the (2×2) matrix identity and

$$\Lambda\left(x,\frac{\partial}{\partial x}\right)=-\frac{e}{2m}\left[2\left(\frac{\partial V}{\partial x}\right)\frac{\partial}{\partial x}+\left(\frac{\partial V}{\partial x}\right)^2\right] \qquad (38)$$

is a supplementary term which depends on the external potential $V(x)$ and does not contribute to the calculation of the effective action.

In effect, using the following formula:

$$\ln\hat{A}=-\int_0^{+\infty}\frac{\mathrm{d}s}{s}\exp(-\mathrm{i}\hat{A}s)+C, \qquad (39)$$

we get

$$\begin{aligned}&\ln\left[\hat{I}\hat{O}_{\mathrm{KG}}+\eta\Lambda\left(x,\frac{\partial}{\partial x}\right)\right]\\&=-\int_0^{+\infty}\frac{\mathrm{d}s}{s}\exp\left\{-\mathrm{i}\left[\hat{I}\hat{O}_{\mathrm{KG}}+\eta\Lambda\left(x,\frac{\partial}{\partial x}\right)\right]s\right\}+\hat{I}C\\&=-\int_0^{+\infty}\frac{\mathrm{d}s}{s}\left[\hat{I}\exp\left\{-\mathrm{i}\left[\hat{O}_{\mathrm{KG}}\right]s\right\}+\eta\hat{R}\right].+\hat{I}C\ , \qquad (40)\end{aligned}$$

where $\hat{R}$ is the operator defined by

$$\hat{R}=\sum_{n=1}^{\infty}\frac{(-\mathrm{i}s)^n}{n!}\left[\sum_{m=1}^{n}\hat{O}^{n-m}\Lambda\left(x,\frac{\partial}{\partial x}\right)\hat{O}^{m-1}\right]. \qquad (41)$$

Being aware of $\mathrm{tr}\eta=0$ and $\mathrm{tr}\hat{I}=2$, we find

$$\begin{aligned}&\mathcal{A}(\text{vac–vac})\\&=\exp\left\{-\mathrm{tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}\left[\exp\left(-\mathrm{i}\hat{O}_{\mathrm{KG}}s\right)+\frac{1}{2}\eta\hat{R}\right]+C\right\}\\&=\exp\left(-\mathrm{tr}\ln\hat{O}_{\mathrm{KG}}\right)=1/\det\hat{O}_{\mathrm{KG}}. \qquad (42)\end{aligned}$$

So the vacuum to vacuum transition amplitude obtained in the FV_0 formulation takes on the same form that we obtain in the KG equation.

3 Bogoliubov transformation method

We have succeeded, in our first approach, to demonstrate that the amplitude $\mathcal{A}$(vac–vac) calculated in the FV_0 formalism is given, like in the KG equation, by the determinant of the operator associated with the wave equation, and we have also shown that the FV_0 two-component formulation which describes spinless particles gives the same amplitude as we obtain in KG theory considering the field ψ_{FV_0} as an ordinary variable.

As a second approach [11–14], we shall use the solutions of the wave equation to determine the probability of pair creation or, explicitly, we shall consider the canonical method based on the Bogoliubov transformation connecting the "in" with the "out" states. Then the probability of pair creation and the probability for the vacuum to remain a vacuum will be given in terms of Bogoliubov coefficients. In other words the relation between "in" and "out" states can be projected in Fock space to be converted into a relation between the creation and annihilation operators; therefore, many physical amplitudes can be calculated easily.

Let us start with the canonical quantization of the FV_0 field.

3.1 FV field quantization

Classically, in order to decompose the FV_0 field to the positive and negative frequency modes, we introduce the Fourier transform $\tilde{\psi}(k)$ of the field $\psi(x)$:

$$\psi(x)=\frac{1}{(2\pi)^2}\int\mathrm{d}^4k\ \mathrm{e}^{\mathrm{i}kx}\tilde{\psi}(k), \qquad (43)$$

which obeys, in the free case,

$$\left[k_0-\frac{\boldsymbol{k}^2}{2m}(\tau_3+\mathrm{i}\tau_2)-m\tau_3\right]\tilde{\psi}(k)=0. \qquad (44)$$

Starting from the solution of the latter equation that we write as

$$\tilde{\psi}(k)=U(k)\ c(k)\ \delta\left(k_0^2-\boldsymbol{k}^2-m^2\right), \qquad (45)$$

where $U(k)$ is a two-component vector and $c(k)$ is a function of variable $(k_0,\boldsymbol{k})$, and after integrating over k^0, one can find

$$\psi(x)=\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left[U(\omega_{\boldsymbol{k}})\,c(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}kx}+V(\omega_{\boldsymbol{k}})\,d(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}kx}\right], \qquad (46)$$

with $\omega_{\boldsymbol{k}}=\sqrt{\boldsymbol{k}^2+m^2}$ and

$$U(\omega_{\boldsymbol{k}})=\frac{1}{\sqrt{4m\omega_{\boldsymbol{k}}}}\begin{bmatrix}m+\omega_{\boldsymbol{k}}\\ m-\omega_{\boldsymbol{k}}\end{bmatrix}, \qquad (47)$$

$$V(\omega_{\boldsymbol{k}})=\frac{1}{\sqrt{4m\omega_{\boldsymbol{k}}}}\begin{bmatrix}m-\omega_{\boldsymbol{k}}\\ m+\omega_{\boldsymbol{k}}.\end{bmatrix} \qquad (48)$$

Let us now quantize the two-component FV_0 field:

$$\psi=\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}, \qquad (49)$$

writing, for a start, the Lagrangian density in terms of the two components φ_1,φ_2:

$$\begin{aligned}\mathcal{L}_{\mathrm{FV}}&=\mathrm{i}\varphi_1^*\dot{\varphi}_1-\mathrm{i}\varphi_2^*\dot{\varphi}_2-m\varphi_1^*\varphi_1-m\varphi_2^*\varphi_2\\&\quad+\frac{1}{2m}\boldsymbol{\nabla}(\varphi_1^*+\varphi_2^*)\,\boldsymbol{\nabla}(\varphi_1+\varphi_2). \qquad (50)\end{aligned}$$

Let P_1 and P_2 be the conjugate moments associated with, respectively, the fields components φ_1 and φ_2:

$$P_i=\frac{\partial\mathcal{L}_{\mathrm{FV}}}{\partial\dot{\varphi}_i},\quad i=1,2. \qquad (51)$$

One can obtain

$$P_1 = \mathrm{i}\varphi_1^*, \;\; P_2 = -\mathrm{i}\varphi_2^* . \tag{52}$$

We assume that at given time t the fields φ_i and their conjugate moments P_i obey the following commutation relation:

$$\left[\varphi_i\left(t,\mathrm{x}\right), P_j\left(t,\mathrm{x}^{'}\right)\right] = \mathrm{i}\delta_{ij}\delta^3\left(\mathrm{x}-\mathrm{x}^{'}\right), \tag{53}$$

which gives for the FV_0 field

$$\left[\varphi_1\left(t,\mathrm{x}\right), \varphi_1^+\left(t,\mathrm{x}^{'}\right)\right] = \delta^3\left(\mathrm{x}-\mathrm{x}^{'}\right), \tag{54}$$

$$\left[\varphi_2\left(t,\mathrm{x}\right), \varphi_2^+\left(t,\mathrm{x}^{'}\right)\right] = -\delta^3\left(\mathrm{x}-\mathrm{x}^{'}\right) \tag{55}$$

and makes the two constants $c\left(\boldsymbol{k}\right)$ and $d\left(\boldsymbol{k}\right)$ that we have defined previously obey

$$\left[c\left(\boldsymbol{k}\right), c^+\left(\boldsymbol{k}'\right)\right] = \left(2\pi\right)^3 2\omega_{\boldsymbol{k}}\delta^3\left(\boldsymbol{k}-\boldsymbol{k}'\right), \tag{56}$$

$$\left[d\left(\boldsymbol{k}\right), d^+\left(\boldsymbol{k}'\right)\right] = -\left(2\pi\right)^3 2\omega_{\boldsymbol{k}}\delta^3\left(\boldsymbol{k}-\boldsymbol{k}'\right). \tag{57}$$

The new operators are interpreted as respectively particle creation and annihilation operators. In the end, using the usual notation,

$$\begin{aligned} c\left(\boldsymbol{k}\right) &\equiv a_{\boldsymbol{k}}, \\ d\left(\boldsymbol{k}\right) &\equiv b_{\boldsymbol{k}}^+, \end{aligned} \tag{58}$$

we obtain the general form of the quantized FV_0 field:

$$\Psi_{\mathrm{op}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{\left(2\pi\right)^3}\left[U\left(\omega_{\boldsymbol{k}}\right)a_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}kx} + V\left(\omega_{\boldsymbol{k}}\right)b_{\boldsymbol{k}}^+\,\mathrm{e}^{-\mathrm{i}kx}\right], \tag{59}$$

which can be generalized, in the presence of an external field, to

$$\Psi_{\mathrm{op}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{\left(2\pi\right)^3}\left[U\left(\omega_{\boldsymbol{k}}\right)a_{\boldsymbol{k}}\;\phi^+\left(x\right) + V\left(\omega_{\boldsymbol{k}}\right)b_{\boldsymbol{k}}^+\;\phi^-\left(x\right)\right], \tag{60}$$

where $\phi^\pm\left(x\right)$ are solutions of the quadratic KG equation,

$$\left[\left(\mathrm{i}\partial_\mu - eA_\mu\right)^2 - m^2\right]\phi\left(x\right) = 0. \tag{61}$$

Note that every time we want to solve the first order FV_0 equation we must turn it into a quadratic form (KG equation) where the solution is the sum of the two components [5].

Calculating the energy E,

$$E = \int \mathrm{d}^3x\;\overline{\psi}H_{\mathrm{FV}}\psi \tag{62}$$

and the electric charge Q

$$Q = e\int \mathrm{d}^3x\;\overline{\psi}\psi, \tag{63}$$

we can give an obvious interpretation of the operators $a_{\boldsymbol{k}}^+$, $a_{\boldsymbol{k}}$, $b_{\boldsymbol{k}}^+$ and $b_{\boldsymbol{k}}$, having

$$E = \int \frac{\mathrm{d}^3\boldsymbol{k}}{\left(2\pi\right)^3 2\omega_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\left[a_{\boldsymbol{k}}^+a_{\boldsymbol{k}} + b_{\boldsymbol{k}}^+b_{\boldsymbol{k}}\right], \tag{64}$$

$$Q = e\int \frac{\mathrm{d}^3\boldsymbol{k}}{\left(2\pi\right)^3 2\omega_{\boldsymbol{k}}}\left[a_{\boldsymbol{k}}^+a_{\boldsymbol{k}} - b_{\boldsymbol{k}}^+b_{\boldsymbol{k}}\right]. \tag{65}$$

Remark that the choice of commutation relations like in KG theory and not anticommutation relations as in the Dirac case will be justified by obtaining a positive probability.

3.2 Pair creation and vacuum instability

For the external potential defined by $\left(A_0\left(z\right),0,0,0\right)$ we solve in the first stage (61) by assuming that $\phi\left(x\right) = \mathrm{e}^{\mathrm{i}\left(\omega t - p_x x - p_y y\right)}\varphi\left(z\right)$; then $\varphi\left(z\right)$ will be a solution of

$$\left[-\frac{\partial^2}{\partial z^2} - \left[\pi_3\left(z\right)\right]^2\right]\varphi\left(z\right) = 0, \tag{66}$$

where

$$\begin{aligned} \pi_3\left(z\right) &= \sqrt{\pi_0^2\left(z\right) - m_\perp^2}, \\ \pi_0\left(z\right) &= \omega - eA_0\left(z\right), \\ m_\perp^2 &= m^2 + p_x^2 + p_y^2. \end{aligned} \tag{67}$$

We classify the behavior of our solutions at $-\infty$ and $+\infty$ as follows:

$$\varphi_\pm\left(z\right) = \frac{1}{\sqrt{2P_-\left(z\right)}}\exp\left(\pm\mathrm{i}\int_0^z P_-\left(z'\right)\,\mathrm{d}z'\right), \tag{68}$$

$$\varphi^\pm\left(z\right) = \frac{1}{\sqrt{2P_+\left(z\right)}}\exp\left(\pm\mathrm{i}\int_0^z P_+\left(z'\right)\,\mathrm{d}z'\right), \tag{69}$$

with

$$\begin{aligned} P_+\left(z\right) &\sim \pi_3\left(z\right)_{z\to+\infty}, \\ P_-\left(z\right) &\sim \pi_3\left(z\right)_{z\to-\infty}. \end{aligned} \tag{70}$$

In all we have four solutions, but there exist two complex coefficients α, β connecting $\varphi_\pm\left(z\right)$ with $\varphi^\pm\left(z\right)$:

$$\begin{aligned} \varphi_+ &= \alpha\varphi^+ + \beta\varphi^-, \\ \varphi_- &= \beta^*\varphi^+ + \alpha^*\varphi^-. \end{aligned} \tag{71}$$

and using the current conservation law we can find that

$$\left|\alpha\right|^2 - \left|\beta\right|^2 = 1. \tag{72}$$

According to Nikishov's choice of "in" and "out" states [12,13] we write

$$\begin{aligned} \varphi_{\mathrm{in}}^+ &= \varphi^-, \\ \varphi_{\mathrm{in}}^- &= -\frac{\beta}{\beta^*}\varphi_-, \\ \varphi_{\mathrm{out}}^+ &= \varphi^+, \\ \varphi_{\mathrm{out}}^- &= \varphi_+, \end{aligned} \tag{73}$$

and from (71) we get

$$\begin{aligned} \varphi_{\mathrm{in}}^+ &= \lambda\varphi_{\mathrm{out}}^+ + \mu\varphi_{\mathrm{out}}^-, \\ \varphi_{\mathrm{in}}^- &= \mu^*\varphi_{\mathrm{out}}^+ + \lambda^*\varphi_{\mathrm{out}}^-, \end{aligned} \tag{74}$$

where $\lambda = -\frac{\alpha}{\beta}$, $\mu = \frac{1}{\beta}$ and $\left|\lambda\right|^2 - \left|\mu\right|^2 = 1$.

Taking into account the total electric charge conservation

$$Q = e\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} \left[a^+_{\mathrm{in},\boldsymbol{k}} a_{\mathrm{in},\boldsymbol{k}} - b^+_{\mathrm{in},\boldsymbol{k}} b_{\mathrm{in},\boldsymbol{k}}\right] = e\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\, 2\omega_{\boldsymbol{k}}} \left[a^+_{\mathrm{out},\boldsymbol{k}} a_{\mathrm{out},\boldsymbol{k}} - b^+_{\mathrm{out},\boldsymbol{k}} b_{\mathrm{out},\boldsymbol{k}}\right], \quad (75)$$

we obtain the following relations:

$$a_{\mathrm{out}} = \lambda\; a_{\mathrm{in}} + \mu^* b^+_{\mathrm{in}}, \quad (76)$$

$$b^+_{\mathrm{out}} = \mu\; a_{\mathrm{in}} + \lambda^* b^+_{\mathrm{in}}. \quad (77)$$

where the index $\boldsymbol{k}$ has been omitted in order to simplify the notation.

For the process of scalar particle pair creation, the probability amplitude that we suggest to calculate is defined by

$$A = \langle 0_{\mathrm{out}} \left| a_{\mathrm{out}} b_{\mathrm{out}} \right| 0_{\mathrm{in}} \rangle . \quad (78)$$

Taking into account (76) and (77) we obtain

$$b_{\mathrm{out}} = \frac{1}{\lambda^*} b_{\mathrm{in}} + \frac{\mu^*}{\lambda^*} a^+_{\mathrm{out}}; \quad (79)$$

then

$$A = \langle 0_{\mathrm{out}} \left| a_{\mathrm{out}} b_{\mathrm{out}} \right| 0_{\mathrm{in}} \rangle = \frac{\mu^*}{\lambda^*} \langle 0_{\mathrm{out}} \mid 0_{\mathrm{in}} \rangle , \quad (80)$$

and the probability to create one pair from the vacuum in state $(\omega, \boldsymbol{p})$ reads

$$\mathcal{P}_{p,\omega} = \left|\frac{\mu}{\lambda}\right|^2 . \quad (81)$$

Using $|\lambda|^2 - |\mu|^2 = 1$, the vacuum persistence is given by

$$\mathcal{C}_{p,\omega} = \left|\frac{1}{\lambda}\right|^2 . \quad (82)$$

Expressions (81) and (82), obtained in the FV_0 formulation, are exactly the same as the ones we obtain in the KG case.

In order to find two-component and eight-component wave functions of the step potential without the use the boundary conditions, the authors of [5,6] have introduced the smooth potential

$$A_0(z) = \frac{V_0}{2}\left(1 + \tanh\frac{z}{2r}\right) \quad (83)$$

as an intermediate stage; it is remarked that the current J is continuous but the charge density ρ is discontinuous:

$$\rho\left(0^+\right) = \frac{\omega - eV_0}{\omega}\rho\left(0^-\right), \quad (84)$$

and the sign of $\rho(0^+)$ may be opposite to the sign of $\rho(0^-)$. In this case the transmitted particle is the antiparticle of the incident one. This means that we have creation of pairs near the step barrier. If T and R denote, respectively, the transmission and reflection coefficients [16,15] we have

$$T = \frac{4p_+p_-}{(p_+ - p_-)^2}, \quad R = \frac{(p_+ + p_-)^2}{(p_+ - p_-)^2}, \quad (85)$$

where, with

$$R - T = 1, \quad (86)$$

the pair creation probability reads

$$\mathcal{P}_{p,\omega} = \frac{T}{R} = \frac{4p_+p_-}{(p_+ + p_-)^2}. \quad (87)$$

We shall demonstrate that the result of Bounames et al. concerning the FV_0 equation makes good sense and it is pertinent with the choice of "in" and "out" states. In effect, we have assumed, with this choice, that the particle in the region $z > 0$ is the antiparticle of the one that moves in the region $z < 0$, still in agreement with the study of Bounames et al. [5,6]. Explicitly, for the potential (83) and the KG equation (66)

$$\left[-\frac{\partial^2}{\partial z^2} - \left(\omega - \frac{eV_0}{2} - \frac{eV_0}{2}\tanh\frac{z}{2r}\right)^2 + m_\perp^2\right]\varphi_z(x) = 0 \quad (88)$$

becomes after changing the variable $z \to \xi$,

$$\xi = \left(1 + \exp\frac{z}{r}\right)^{-1}, \quad (89)$$

a hypergeometric type equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \left(\frac{1}{\xi} - \frac{1}{1-\xi}\right)\frac{\mathrm{d}}{\mathrm{d}\xi} + \left(\frac{(p_+r)^2}{\xi} - (eV_0r)^2 + \frac{(p_-r)^2}{1-\xi}\right)\frac{1}{\xi(1-\xi)}\right] f(\xi) = 0, \quad (90)$$

which is invariant under the change $\xi \to 1-\xi$ and $p_+ \to p_-$ and admits two solutions [17]:

$$f_1(\xi) = N\, \xi^{\mathrm{i}p_+r}\, (1-\xi)^{\mathrm{i}p_-r} \times F(\mathrm{i}p_+r + \mathrm{i}p_-r + \delta, \mathrm{i}p_+r + \mathrm{i}p_-r + 1 - \delta, 2\mathrm{i}p_+r + 1; \xi) \quad (91)$$

and

$$f_2(\xi) = N\, \xi^{\mathrm{i}p_+r}\, (1-\xi)^{\mathrm{i}p_-r} \times F(\mathrm{i}p_+r + \mathrm{i}p_-r + \delta, \mathrm{i}p_+r + \mathrm{i}p_-r + 1 - \delta, 2\mathrm{i}p_-r + 1; 1-\xi), \quad (92)$$

where

$$\delta = \frac{1}{2}\left[1 - \sqrt{1 - (2eV_0r)^2}\right]. \quad (93)$$

The functions $\varphi^+_{\mathrm{in}}, \varphi^-_{\mathrm{in}}, \varphi^+_{\mathrm{out}}$ and φ^-_{out} are then respectively

$$\varphi^+_{\mathrm{in}} = \frac{1}{\sqrt{2p_+}}\exp(-\mathrm{i}p_+z), \quad (94)$$

$$\varphi_{\text{in}}^{-}=-\frac{\beta}{\beta^{*}}\frac{1}{\sqrt{2p_{-}}}\exp\left(-\mathrm{i}p_{-}z\right),\tag{95}$$

$$\varphi_{\text{out}}^{+}=\frac{1}{\sqrt{2p_{+}}}\exp\left(\mathrm{i}p_{+}z\right),\tag{96}$$

$$\varphi_{\text{out}}^{-}=\frac{1}{\sqrt{2p_{-}}}\exp\left(\mathrm{i}p_{-}z\right).\tag{97}$$

With the help of the formula

$$F\left(a,b,c;y\right)=A_{1}F\left(a,b,a+b-c+1;1-y\right)\tag{98}$$
$$+A_{2}\left(1-y\right)^{c-a-b}F\left(c-a,c-b,c-a-b;1-y\right),$$

where

$$A_{1}=\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)},$$
$$A_{2}=\frac{\Gamma\left(c\right)\Gamma\left(a+b-c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)},\tag{99}$$

both of the coefficients α and β can be determined:

$$\alpha=\sqrt{\frac{p_{-}}{p_{+}}}\tag{100}$$
$$\times\frac{\Gamma\left(-2\mathrm{i}p_{+}r+1\right)\Gamma\left(-2\mathrm{i}p_{-}r\right)}{\Gamma\left[\frac{1}{2}-\mathrm{i}\left(p_{+}+p_{-}\right)r-\frac{\delta}{2}\right]\Gamma\left[\frac{1}{2}-\mathrm{i}\left(p_{+}+p_{-}\right)r+\frac{\delta}{2}\right]},$$

$$\beta=\sqrt{\frac{p_{-}}{p_{+}}}\tag{101}$$
$$\times\frac{\Gamma\left(-2\mathrm{i}p_{+}r+1\right)\Gamma\left(2\mathrm{i}p_{-}r\right)}{\Gamma\left[\frac{1}{2}+\mathrm{i}\left(p_{+}+p_{-}\right)r-\frac{\delta}{2}\right]\Gamma\left[\frac{1}{2}+\mathrm{i}\left(p_{+}+p_{-}\right)r+\frac{\delta}{2}\right]},$$

and one can find then

$$\mathcal{P}_{p\omega}=\frac{p_{+}}{p_{-}}$$
$$\times\left|\frac{\Gamma\left[\frac{1}{2}-\mathrm{i}\left(p_{+}+p_{-}\right)r-\frac{\delta}{2}\right]\Gamma\left[\frac{1}{2}-\mathrm{i}\left(p_{+}+p_{-}\right)r+\frac{\delta}{2}\right]}{\Gamma\left(-2\mathrm{i}p_{+}r+1\right)\Gamma\left(-2\mathrm{i}p_{-}r\right)}\right|^{2}$$
$$=\frac{2\sinh\left(2\pi p_{+}r\right)\sinh\left(2\pi p_{-}r\right)}{\cosh\left[2\pi\left(p_{+}+p_{-}\right)r\right]+\cos\left(\pi\omega\right)}.\tag{102}$$

Taking the limit $r\to 0$ the potential barrier becomes a step one, and we obtain the well-known result

$$\mathcal{P}_{p\omega}=\frac{4p_{+}p_{-}}{\left(p_{+}+p_{-}\right)^{2}}.\tag{103}$$

Note that by taking the limit $r\to\infty$ with $V_0/r=E$ = constant and by the use of the approximation

$$p_{+}\simeq eV_{0}-\omega-\frac{m_{\perp}^{2}}{2eV_{0}}\tag{104}$$

we get

$$\mathcal{P}_{p\omega}=\frac{\exp\left(-\pi\frac{m_{\perp}^{2}}{eE}\right)}{1+\exp\left(-\pi\frac{m_{\perp}^{2}}{eE}\right)},\tag{105}$$

which is the probability of pair creation from the vacuum by a constant electric field.

4 Conclusion

We have studied the behavior of the FV_0 equation in the problem of particle pair creation from the vacuum, considering two approaches.

With the approach of the Schwinger effective action, treating the two-component field ψ_{FV_0} like a bosonic field we have shown that the FV_0 equation gives the same result as the KG one. The attempt to treat ψ_{FV_0} as a fermionic field, because of the resemblance of the FV_0 and the Dirac equation, which are two first order and matrix equations, leads to non-physical probabilities (negative probabilities).

With the second approach which uses the canonical quantization, the calculation of probabilities with FV_0 necessitates knowledge of the states “in” and “out” for the KG field. Then the results with FV_0 and KG become identical. Therefore, we have used commutators for FV_0 fields as in the KG one and not the anticommutators that will also lead to erroneous probabilities.

We conclude, through the problem of particle pair creation, that the FV_0 and KG equations have the same behavior vis-à-vis physical processes and, consequently, using FV_0 or KG, we obtain the same result. This final conclusion is the same as that communicated by the authors of [7] in a study of the Compton effect, but with the spin $\frac{1}{2}$ eight-component equation ($\mathrm{FV}_{\frac{1}{2}}$) and that of Dirac.

The study of spin $\frac{1}{2}$ particle creation with the $\mathrm{FV}_{\frac{1}{2}}$ eight-component formulation is under consideration. The result will be published elsewhere.

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